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Research Report CCS 368

(1) THE MDI METHOD AS A GENERALIZATION OF LOGIT, PROBIT AND HENDRY ANALYSES IN MARKETING.

by

(10) A. Charnes*
W.W. Cooper**
D.B. Learner***
F.Y. Phillips***

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*The University of Texas at Austin
**Harvard University
***Market Research Corporation of America

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CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director
Business-Economics Building, 203E
The University of Texas at Austin
Austin, TX 78712
(512) 471-1821

ABSTRACT

Constrained minimum discrimination information methods provide a basis for a unified approach to a wide range of problems in marketing research. For instance, they lead to characterizations parallel to those of the Hendry system and other entropic approaches, with greater economy of assumptions. Goodness-of-fit tests and a structure for decision modelling are supplied from the same basic models with a range of applications that include market segmentation and brand shifting choices. Other probabilistic models of marketing choice (logit, MCI, etc.) are also comprehended in ways that resolve many logical and computational difficulties in these other approaches.

KEY WORDS

Marketing Models

Information Theory

Statistics

Individual Choice Models

Unifying Market Research

Market research has become increasingly complex. This characterization applies to methods of analysis as well as their areas of application. An effort at unifying these proliferating tools, techniques and concepts would thus seem to be in order.

It is the purpose of this paper to provide a basis for such unification in ways which also serve to increase the power of these disparate developments. Thus, the proposed methodological (and conceptual) unification is to be attained in a way that will allow these various developments to continue, when they are applicable, but also to provide an underlying conceptual-methodological framework with which to relate them to each other.

Our approach to the proposed unification will be via "information theory". In particular, we shall use what is called the "information statistic" and show how different models and methods which are commonly used in various parts of marketing can be related to this one statistic. In particular we shall show how both decision theoretic as well as classical statistical methods can thereby be related. The latter will include classical techniques of regression and correlation, as used in marketing, as well as more recent variants such as "logit" and "probit" analysis, etc. It follows that a single consistent basis will thereby also be supplied for unifying the research approaches to different marketing areas such as brand switching, market segmentation, store location and market areas, etc. Conversely the availability of an efficient unified approach, such as we will be suggesting, will also supply a basis for reviewing past results in the light of new alternatives that will thereby be brought into view.

Unifying Statistics

As far back as 1925, R.A. Fisher* advanced the basic notion that the discipline of statistics could be regarded as being concerned with information and its measurement. For the normal distribution with which he was then concerned, Fisher showed that the reciprocal of the variance provides a measure of the average amount of information supplied by each unit in a sample for estimating the corresponding population mean.** Thus, the observations in a sample with a large variance communicate less information (per observation) than would be the case for a sample with a smaller variance.

Drawing upon concepts from classical physics, Claude Shannon and Norbert Wiener*** (circa 1949) developed a measure of information for messages connmunicated in the form of binary digits (BITS) as in, for instance, a modern digital computer. Theirs was a probability based approach, however, and therefore seemed to differ from Fisher's statistics-based approach. Kullback and Leibler in 1951, however, took a different tack.**** They developed the statistical properties of this information measure and showed the applicability of what is now called the "Kullback-Leibler statistic" to a wide variety of statistical problems and methods --including the devlopments of R.A. Fisher.

This work by Kullback (1959) and by Kullback and Leibler (1951) also supplied a basis for still further progress. Akaike (1973)*****,

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*See Fisher (1935).

**We are here giving only a rough characterization of Fisher's expression of his thoughts. For full detail, see Fisher (1935).

***See Shannon and Weaver (1949). See also Khinchine (1957).

****These topics are discusses in Kullback (1959).

*****See Akaike (1973), (1977) and (1978). See also Sawa (1978).

for instance, was able to show that the Kullback-Leibler statistic could be used to unify supposedly separate parts of statistics such as decision theory and classical (maximum likelihood) approaches. He was also able to resolve a variety of paradoxes and to deal with open questions such as the number of terms to include in a regression or a factor analysis in a precise statistical manner.

This all suggests that the end of these developments is still not in sight. It also suggests that our proposed basis for unification will better position different parts of market research to take advantage of these developments as they occur. In any case the task of the immediately following sections will be to exhibit how the proposed unification might now be achieved. In addition we will exploit the recent results of A. Charnes, W.W. Cooper, et. al.* which brings together information theory and mathematical programming to deal with policy evaluation and statistical inference in a single model. This makes it possible to study marketing plans and management policies and to evaluate their consequences in ways that would not previously have been possible.

Although the sections that follow will require mathematical notation, we will supply references rather than formal proofs. After this has been done we will return to purely verbal characterizations and interpretations in the concluding section of this paper. This will allow us to summarize what we will have covered and to indicate possible courses of further development. Here, however, we need to emphasize that only a beginning has thus far been made in the proposed unification and much more remains to be done.

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*See Charnes and Cooper (1974); Charnes, Cooper, and Learner (1978); Charnes, Cooper, and Seiford (1978); Brockett, Charnes, and Cooper (1978); Phillips (1980).

The MDI Method

The Kullback-Leibler statistic may be written

$$I(p:q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}$$

Its constrained minimum, called the Minimum Discrimination Information (MDI) statistic, gives the information in favor of the distribution $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$ as against the distribution $q_i \geq 0$, $\sum_{i=1}^{n} q_i = 1$. We elaborate this further as follows.

Minimizing the information I(p:q) for discrimination between the probability distributions p and q, subject to any constraints that may apply to the parameters of p, results in an estimate of p which is the distribution least distinguishable from q, but which satisfies the constraints (which q itself may not do). In many important cases, these information theoretic estimates are maximum likelihood estimates, and they are in general best asymptotically normal.*

An asymptotic distribution theory of I*(p:q) (the minimum discrimination information (MDI) value) leads to a test of the hypothesis that p and q are identical, i.e. that the observed parameters are consistent with the estimated parameters. Estimation and hypothesis testing are thus achieved simultaneously. Since I(p:q) is a general measure of the "distance" between p and q, all estimates and inferred relationships resulting from a constrained MDI problem are valid whether or not H_0 is accepted.
Noting the above, Charnes, Cooper and Learner (1978) brought an extended version of MDI to the problem of brand shifting as incorporated in MCRA's SANDDABS model. This extended version comprehends an approach to MDI under

*See Gokhale and Kullback (1978a) for a full discussion.

inequality constraints with a new duality relation involving an especially simple unconstrained convex functional which can be used to provide additional insights (and power) to MDI approaches and also to simplify the computations.*

This also makes it possible to regard the following commonly used marketing models as special cases of the general MDI model: the loglinear model, the maximum entropy model, logit and probit models, as well as gravity models and "multiplicative competitive interaction" models of individual choice, and others. We will detail some of these relationships in summary fashion as follows: First, we will relate the MDI to Bayes' theorem from which its relation to decision theory will be evident. From the canonical MDI model we will then derive each of the implied marketing models and indicate how the MDI hypothesis testing capability makes these models more useful for management decision making. For the sake of brevity, the demonstrations are confined to the simplest case of each subsidiary model (for instance the univariate dichotomous logit model), but extensions are indicated, as well as new variants of the basic models suggested by the MDI framework.

*Further progress includes the derivation of characterizations of the complete duality states (Charnes, Cooper and Seiford (1978)) and relations to other types of statistical analyses as in Brockett, Charnes and Cooper (1978).

MDI, Divergence and Bayes' Theorem

Following Kullback (1959, p.4), we first relate the MDI statistic to Bayes' theorem (and hence to decision theory) by writing

$$p(x) = P(x|H_1)$$

$$q(x) = Q(x|H_2)$$

where p(x) and q(x) are statistical distributions with components p_i, $q_i \geq 0$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$.

Here H_1 and H_2 represent hypotheses which associate the sample values X = x with P and Q respectively. By Bayes' theorem,

$$\frac{P(H_1|x)}{Q(H_2|x)} = \frac{P(x|H_1)P(H_1)}{Q(x|H_2)Q(H_2)} = \frac{p(x)P(H_1)}{q(x)Q(H_2)}$$

or

$$\ln \frac{p(x)}{q(x)} = \ln \frac{P(H_1|x)}{Q(H_2|x)} - \ln \frac{P(H_1)}{Q(H_2)}$$

The expression on the left is the "log-odds" ratio for the distribution p against the distribution q on the basis of X = x. It is evidently the difference between the posterior and prior distributions in terms of the logarithms of their ratios.

Thus the statistic

$$I(p:q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}$$

represents the expected value of this gain. Furthermore we can also introduce the statistic which Kullback (1959) refers to as the "divergence

measure", via

$$J(p,q) = I(p:q) + I(q:p)$$
$$= \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} + \sum_{i=1}^{n} q_i \ln \frac{q_i}{p_i}$$
$$= \sum_{i=1}^{n} (p_i - q_i) \ln \frac{p_i}{q_i} ,$$

which represents a generalization of the usual "generalized distance" statistic of Mahalanobis. In these terms I(p:q) and I(q:p) represent what might be called "directed divergences" and J(p,q) a measure of the divergence between H_1 and H_2 on the basis of X = x. The latter, i.e. J(p,q) has all the properties of a distance measure, except that it need not satisfy the triangle inequality.* The quantity I(p:q) may be referred to as the Kullback-Leibler statistic or the DI (discrimination information) statistic in that we have not yet introduced the minimization principle for selecting p and q.

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*See Appendix A in Charnes and Cooper (1961).

The Canonical MDI Problem

We refer to problems of the following form as "canonical":

$$\text{Minimize} \quad I(p:q) \equiv \sum_{j=1}^{n} p_j \ln(p_j/q_j)$$

$$\text{subject to} \quad \sum_j a_{ij} p_j = \theta_i \qquad i=1,\ldots,m$$

$$\sum_j p_j = 1$$

$$\text{all} \quad p_j \geqslant 0$$

--The q_j and θ_i are constants. The q_i may be hypothesized values and the θ_i sample statistics; or vice versa. In either case, the implied null hypothesis is H_0:p=q, i.e. that the observed and expected figures are not distinguishable.

--Denote I*(p:q) as the solution for which p=p*, the minimizing choice of p. Then 2NI*(p:q) (N is the sample size) is asympototically distributed as chi-square, with degrees of freedom depending on n and on the number of linearly independent constraints (see Gokhale and Kullback (1978), Phillips (1980)).

MDI and the Loglinear (Multiplicative) Model

Using the method of Lagrange (undetermined) multipliers, Gokhale and Kullback (1978) prove that the solution of the canonical MDI problem leads to the loglinear representation of the estimates p_j^*:

$$\ln p_j^* - \ln q_j = \ln \frac{p_j^*}{q_j} = \sum_{i=1}^{m} \lambda_i^* a_{ij}$$

where the λ_i^* are determined to satisfy the constraints. This is the loglinear model as described in, e.g. Bishop, Fienberg and Holland (1975).

--The p_j^* in this loglinear model automatically sum to one, since this is a condition of the MDI problem.

--Gokhale and Kullback stress that this loglinear model of the p_j^* is a consequence of the MDI formulation, and is not derived from seemingly arbitrary assumptions of convenience as in, e.g. Jones and Zufryden (1978).

--The term $\ln q_j$ in the log-odds equation is not merely a constant of fit. Its meaning is implied by the derivation of I(p:q) via Bayes' theorem in the earlier section.

--An exponential transformation of both sides of the loglinear equation yields the multiplicative equation variant,

$$p_j^* = q_j \exp\left\{\sum_{i=1}^{m} \lambda_i^* a_{ij}\right\} = q_j \prod_{i=1}^{m} e^{\lambda_i^* a_{ij}}$$

--The loglinear model has application in several aspects of marketing research (Green, Carmone and Wachspress (1977)); in transportation research (Oum (1979), Phillips (1978), McFadden (1973)); in representing production functions (Charnes, Cooper and Schinnar (1976)); and physical systems (see Phillips (1978)). Later sections of this paper link the loglinear model to logit and MCI models, where further marketing applications are cited.

The Dual Convex Programming Form of the Constrained MDI

Charnes, Cooper and Seiford (1978) proved the complete mathematical programming duality theory for constrained MDI estimation, in terms of the following dual problems:

primal	dual
$\sup v(\delta) \equiv -\sum_i \delta_i \ln\left(\frac{\delta_i}{ec_i}\right)$	$\inf \xi(z) \equiv \sum_i c_i e^{{}_iAz} - b^T z$
$A^T\delta = b$	z unconstrained.
$\delta \geqslant 0$	

--Here ${}_iA$ denotes i^{th} row of A.

--The duality state of interest *comprehends the conditions*:

(i) a feasible δ exists with every $\delta_i > 0$;

(ii) $\xi(z)$ has a minimum at z^* and $v(\delta)$ has a unique maximum at δ^*;

(iii) $\xi(z^*) = v(\delta^*)$; and

(iv) $\delta_i^* = c_i e^{{}_iAz^*}$.

See Brockett, Charnes and Cooper (1978) for a complete statement. Note that condition (iv) is the multiplicative (loglinear) model of the estimates δ_i^*.

--The estimates are easily computed by minimizing the unconstrained convex function $\xi(z)$ then transforming z^* to δ^* by means of formula (iv).

MDI and the Entropy Model

Max-entropy models have become well-known in transportation research and more recently in marketing research (see Herniter (1973, 1974); and Phillips (1978) for additional references). The discrete entropy model involves maximizing the "entropy" function of a distribution p:

$$\text{Maximize } H(p) = -\sum_{j=1}^{n} p_j \ln p_j$$

subject to linear constraints. It is easily seen that H(p) finds its extremum at the same point as does I(p:q) if we let $q_j = 1/n$ for $j=1,\ldots,n$:

$$\begin{aligned} I(p:q) &= \sum_{j=1}^{n} p_j \ln(p_j n) \\ &= \sum p_j \ln p_j + \ln n \\ &= -H(p) + (\text{constant}). \end{aligned}$$

H(p) and I(p:q) are therefore measures of the deviation of p from a discrete uniform distribution over n points; however in this regard the entropy function is clearly a special case of the discrimination information statistic.

--The MDI is more general and offers greater flexibility, since the null-hypothesis function q can represent any probability function, (not just a uniform distribution).

--The MDI theory has a complete and rigorous foundation in statistics, so we need not be troubled by non-rigorous analogies from thermodynamics--as is so often the case with "entropic" models (see Phillips (1978), Haynes, Phillips and Mohrfeld (1980)).

MDI and the Logit Model

The logit model is a special case of the loglinear model (see Green, Carmone and Wachspress (1977, p. 56)). The logit uses a linear function of several independent variables x_i to represent the log-odds of the occurrence of an event E, given a value of x.

$$\ln \frac{P(E)}{1-P(E)} = \sum_{i=1}^{n} \beta_i x_i$$

We consider below the dichotomous case (occurrence/nonoccurrence of E) with one independent variable. An example from Berkson (1972) and Gokhale and Kullback (1978) uses the following four samples under different values of x:

Sample #	Value of x	Sample Size	# of Successes
1	0	10	1
2	1	10	6
3	2	10	3
4	3	10	8
		40	18

Transform to a contingency table representation:

x	i		Success (j=1)	Failure (j=2)	Σ
0	1		1	9	10
1	2		6	4	10
2	3		3	7	10
3	4		8	2	10
		Σ	18	22	

Solve the max-entropy problem:

$$\text{Max} \quad H(p_{ij}) = -\sum_i \sum_j p_{ij} \ln p_{ij}$$

subject to: $$\sum_{j=1}^{2} p_{ij} = 10 \qquad i=1,2,3,4$$

$$\sum_{i} p_{i1} = 18$$

$$\sum_{i} x_i p_{i1} = 36$$

The loglinear model results:

$$\ln p^*_{11} = x_1 z^*_6 + z^*_5 + z^*_1 \quad ;$$

$$\ln(1-p^*_{11}) = \ln p^*_{12} = z^*_1$$

This implies $\ln\left[\frac{p^*_{11}}{1-p^*_{11}}\right] = z^*_6\, x_1 + z^*_5$.

The loglinear representations of the remaining p_{i1}, p_{i2} have the same respective coefficients, thus

$$\ln\left[\frac{P(E|x)}{1-P(E|x)}\right] = z^*_6 x + z^*_5$$

as required in the simple logit model, and $P(E|x)$ is given by the logistic cumulant function

$$P(E|x) = [1+e^{-(z^*_6 x + z^*_5)}]^{-1}$$

We now consider the MDI extension of this max-entropy-to-logit sequence. If above we replace Max $H(p_{ij})$ by Min $I(p_{ij}:q_{ij})$, the resulting log-odds are

$$\ln \frac{p_{ij}}{1-p_{ij}} = \ln \frac{q_{ij}}{1-q_{ij}} + z^*_5 + z^*_6 x.$$

Evidently if the prior log-odds $\ln\,(q_{ij}/1-q_{ij})$ is a linear function of x, then $\ln(p_{ij}/1-p_{ij})$ will also be a linear function of x. Otherwise, the resulting representation will constitute a nonlinear generalization of the logit model.

The trichotomous univariate case (three alternatives, one independent variable) can be handled similarly. Again using the contingency table representation

x	i	j:	1	2	3	Σ
0	1		1	9	5	15
1	2		7	2	6	15
2	3		1	3	11	15
3	4		8	2	5	15
		Σ :	17	16	27	

we minimize $I(p_{ij} : q_{ij})$

subject to $$\sum_{j=1}^{3} p_{ij} = 15 \qquad i = 1,2,3,4$$

$$\sum_{i=1}^{4} p_{i1} = 17$$

$$\sum_{i=1}^{4} p_{i2} = 16$$

$$\sum_{i=1}^{4} x_i p_{i1} = 33$$

$$\sum_{i=1}^{4} x_i p_{i2} = 14$$

$$p_{ij} \geqslant 0 \qquad \forall\, i,j$$

Taking ratios of the loglinear representations as we did in the dichotomous case, we have that for every value of x,

$$\ln \frac{p_1^*}{p_2^*} = \ln \frac{q_1}{q_2} + (z_7^* - z_8^*)x + (z_5^* - z_6^*);$$

$$\ln \frac{p_1^*}{p_3^*} = \ln \frac{q_1}{q_3} + z_7^*x + z_5^* ;$$

and $$\ln \frac{p_2^*}{p_3^*} = \ln \frac{q_2}{q_3} + z_8^*x + z_6^* .$$

The three inverse logits are obtained by reversing signs, since ln x/y is equal to -ln y/x.

We illustrate one more case below, that of the dichotomous logit model with two independent variables x and y. For notational clarity we allow x to take on four values and y three values in this example, which can therefore be visualized as a three-way table $\left[p_{ijk}\right] = \left[\text{number of times alternative } j \text{ is chosen when } x = x_i \text{ and } y = y_k\right]$. In the MDI constraint set below, we have replaced the right-hand-sides with symbols, since explicit values are not necessary for the logit derivation.

$$\text{Min } I(p_{ijk} : q_{ijk})$$

$$\text{s.t.} \quad \sum_{j=1}^{2} p_{ijk} = r \qquad \begin{array}{l} i = 1,2,3,4 \\ k = 1,2,3 \end{array}$$

(sample size under each of the 12 (x,y,) combinations)

$$\sum_{i=1}^{4} \sum_{k=1}^{3} p_{i1k} = r_{13}$$

(total number of observed choices of the first alternative)

$$\sum_{i=1}^{4} x_i \sum_{k=1}^{3} p_{i1k} = r_{14}$$

(expected value of x given that the first alternative is chosen)

$$\sum_{k=1}^{3} y_k \sum_{i=1}^{4} p_{i1k} = r_{15}$$

(expected value of y given that the first alternative is chosen)

Then, for any i and k,

$$\ln \frac{p_{i1k}}{p_{i2k}} = \ln \frac{p_{i1k}}{1-p_{i1k}} = \ln \frac{q_{i1k}}{q_{i2k}} + z^*_{15} y_k + z^*_{14} x_i + z^*_{13}$$

that is, $$\ln \frac{P(E)}{1-P(E)} = \ln \frac{Q(E)}{1-Q(E)} + z^*_{15} y + z^*_{14} x + z^*_{13}$$

---It is straightforward to combine the latter two cases for a general polytomous (multinomial) multivariate logit model, e.g. the one given by Theil (1969):

$$\log \frac{P_i}{P_j} = (\alpha_i - \alpha_j) + \sum_k \gamma_k \left[\log y_{ki} - \log y_{kj}\right] + \sum_h (\beta_{hi} - \beta_{hj}) \log x_h.$$

In Theil's model the y variables are levels of attributes of the choice objects (e.g. price, nutrition, % distribution, advertising exposure, etc.), and the x variables are measures of consumer characteristics (e.g. income, number of children, purchase histories, etc.). This is consonant with the MDI derivation of the logit model.

--McFadden (1973) presents a utility-derived variant of the multinomial multivariate logit which gives a "conditional logit" expression for the choice odds given the consumer characteristics. The conditional choice probabilities in McFadden's model can be separated and written as

$$P_i = P(x_i|s) = e^{\sum_{k=1}^{k} z_i^k \Theta^k} \Big/ \sum_{j=1}^{J} e^{\sum_{k=1}^{k} z_j^k \Theta^k} ,$$

where s is a vector of consumer characteristics and $\sum_{k=1}^{k} z_i^k \Theta^k$ *is the* linear "utility function" of s-type consumers for alternative i. The Θ^k are unknown. We see that

$$P_i = \prod_{k=1}^{k} e^{z_i^k \Theta^k} \Big/ \sum_{j=1}^{J} \prod_{k=1}^{k} e^{z_j^k \Theta^k} = \prod_{k=1}^{k} (y_i^k)^{\Theta k} \Big/ \sum_{j=1}^{J} \prod_{k=1}^{k} (y_j^k)^{\Theta k}$$

under the transformation $y_j^k = e^{z_i^k}$. This shows McFadden's conditional logit to be identical to the "MCI" model dealt with in a later section of this paper, and estimable in the same manner using the MDI method.

--Theil's empirical use of the logit model and McFadden's utility - theoretic derivation are both to be contrasted to the MDI approach, in which the logit expressions follow from the solution of a structural, or "process" model, which explicitly represents all of the known information relevant to the choice situation.

Our logit examples should make it apparent that the form of the logit or loglinear representation depends on the structure of the MDI constraints, which depend in turn on the kind of information that is available for constructing the model. However, it is often the case that a given set of structural (or policy) conditions can be represented by many distinct but equivalent sets of linear equations.* Thus within the MDI format as elsewhere, the available information may suggest, but will not determine, the form of the estimation model. Recasting the constraints in the above examples, for instance, may produce some of the logit variants mentioned by Oum (1979). We will not pursue this possibility here.

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*even by many equivalent sets of linearly independent linear equations, in many cases--although linear independence of the constraints is not a prerequisite for an MDI solution under the Charnes-Cooper theory.

The substitution of equivalent constraint sets will not affect the value of $I^*(p:q)$ for a given problem (Gokhale and Kullback (1978)), but may affect the significance values of individual constraints (see Phillips (1980) for a detailed discussion).

--It is possible that this more comprehensive framework for the logit model will resolve some of the problems that arise in its application (see e.g. Oum (1979)).

--Marketing applications of logit models are due to Green, Carmone and Wachspress (1977), Jones and Zufryden (1978, 1979), Flath and Leonard (1979) and McFadden (1973). The latter uses the "conditional" logit in a more general context of choice behavior.

MDI and the Probit Model

(Dichotomous case with one independent variable)

This simple probit model involves nothing more than fitting a normal distribution function (Hanushek and Jackson (1977)). For a given value of x, the probit model represents the probability of occurrence of an event E as

$$P(E|x) = \Phi(\beta x) \equiv P(Y \leqslant \beta x)$$

where Y is a standard normal variate.

Then $\frac{d}{dx}P(E|x) = \varphi(\beta x)$ where φ is the standard normal density function.

We entertain the hypothesis H_0: $P(E|x) = \Phi(\bar{\beta}x)$.
It is sufficient to test

$$H_0': \frac{d}{dx}P(E|x) = \varphi(\bar{\beta}x).$$

Suppose we observe a parameter $\theta = \int T(x)\, dP(E|x)$, and solve the MDI:

$$\text{Min} \int_{-\infty}^{\infty} f(x) \ln \frac{f(x)}{\varphi(\bar{\beta}x)} dx \qquad (*)$$

$$\text{s.t.} \int_{-\infty}^{\infty} T(x)\, f(x) = \theta,$$

where we let $f(x) = \frac{d}{dx} P(E|x)$. In Khinchin's (1957) terminology, the $f^*(x)$ solving this problem is the "conjugate distribution" of $\varphi(\bar{\beta}x)$. Kullback's (1959) theorem on the MDI inequality implies that if

$M(z) \equiv \int e^{zT(x)}\varphi(\bar{\beta}x)\lambda(dx)$ exists on an interval, and if $f^*(x)$ solves problem $(*)$, then (and only then) we have

$$f^*(x) = \frac{e^{zT(x)}\,\varphi(\bar{\beta}x)}{M(z)}, \qquad \text{and}$$

$$I\left[f^*(x) : \varphi(\bar{\beta}x)\right] = I^* = \theta z - \ln M(z), \text{ where } \theta = \frac{d}{dz}\ln M(z).$$

For example, let $\theta = E(x)$. Then $T(x) = x$, and for problem $(*)$ above, the theorem implies

$$f^*(x) = \frac{e^{xz}\left\{(2\pi)^{-\frac{1}{2}}\,e^{-x^2/2}\right\}}{(2\pi)^{-\frac{1}{2}}\int e^{-x^2/2}\,e^{xz}\,dx} = \frac{e^{-x^2/2+xz}}{\int e^{-x^2/2+xz}\,dx}.$$

Completing the square in the exponents,

$$f^*(x) = \frac{e^{-\frac{1}{2}(x-z)^2}\,e^{\frac{1}{2}z^2}}{e^{\frac{1}{2}z^2}\int e^{-\frac{1}{2}(x-z)^2}\,dx} = \frac{e^{-\frac{1}{2}(x-z)^2}}{\sqrt{2\pi}},$$

which is the normal density function with mean z and unit variance. Thus our estimate of $P(E|x)$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\bar{\beta}x} e^{-\frac{1}{2}(t-z)^2}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\bar{\beta}x-z} e^{-\frac{1}{2}t^2}\,dt$$

$$= \Phi(\bar{\beta}x - z)\;;$$

i.e. the conjugate distribution of a normal distribution is another normal

distribution. We may generalize for other Θ. The parameter(s) Θ may be assumed observable in the same manner as in the discussion of the logit model.

--Applications of logit and probit models have been hindered by problems such as the "independence of irrelevant alternatives" assumption (McFadden (1973), Hausman and Wise (1973)). One way of stating this assumption (McFadden (1973)) is that following the introduction of a new alternative into the choice set, the new share of old alternative i, p_i^{new}, should be equal to $(1-m)p_i^{old}$, where m is the stable share of the new alternative and p_i^{old} was the share of alternative i prior to the introduction of the new alternative. If it is possible to sample the distribution of choices both before and after this introduction, the irrelevant alternatives assumption can be tested via MDI or the Pearson chi-square. The advantages of MDI in this regard were intimated by Theil (1969). The work of Charnes, Cooper, Learner and Phillips (1980) is also relevant for determining whether an alternative is vulnerable to share loss to another member of the universe of choices. The question of relevant alternatives changes with the product life cycle and the point of view of the investigator. During the growth period of carbonated soft drinks' share of the total beverage market, coffee was the alternative of interest in studies conducted by soft drinks trade associations (Woodruff and Phillips (1974)). These days, with a stable category share, switching studies are sponsored by individual manufacturers, and focus on preference shifts within the soft drink category.

--For applications of probit models relevant for marketing, see Hausman and Wise (1978) and Hanushek and Jackson (1977). See also Daganzo (1979).

MDI and MCI

Nakanishi and Cooper (1974) set forth the "Multiplicative Competitive Interaction" model of individual choice:

$$p_j = \frac{\prod_k x_{kj}^{\beta_k}}{\sum_j \prod_k x_{kj}^{\beta_k}}$$

as a generalization of an earlier choice model due to Huff (1962). The probability of choosing alternative j is given as a (normalized) product of terms. Each term reflects the amount of attribute k carried by alternative j, raised to the exponent for attribute k.

We now derive this model via an entropic principle.

$$\text{Max} \quad - \sum_j v_j \ln v_j$$

$$\text{s.t.} \qquad \sum_j v_j x_{kj} = T_k \qquad \forall\, k$$

$$v_j \geq 0 \qquad \forall\, j$$

This problem is stated in terms of a frequently purchased consumer good, e.g. a packaged food. v_j is the number of pounds (or packages) of brand j purchased in a specified time interval. x_{kj} is now taken to be the amount of attribute k per package or per pound of brand j. T_k represents the total amount of attribute k consumed during the interval by the population under

study. (This quantity can be enumerated from consumer panel data for certain kinds of attributes.)

At optimum, we have (see the earlier discussion of loglinear models):

$$v_j^* = \exp\left[\sum_k x_{kj}\, z_k^*\right] = \prod_k e^{x_{kj} z_k^*}$$

$$= \prod_k (e^{x_{kj}})^{z_k^*}$$

$$= \prod_k y_{kj}^{z_k^*}$$

where the z_k^* are dual evaluators, and in the last expression we have substituted y_{kj} for $e^{x_{kj}}$.

It is then straightforward that the probability Prob $\{$a package purchased will be a package of brand j$\}$ can be written

$$p_j = \frac{v_j^*}{\sum_j v_j^*} = \frac{\prod_k y_{kj}^{z_k^*}}{\sum_j \prod_k y_{kj}^{z_k^*}} ,$$

which is the MCI model.

--Extensions can be made in the MDI constraint set to accomodate attributes that cannot be expressed on a per-pound or per-package basis.

--McFadden's conditional logit model, mentioned in an earlier section, illustrates the relationship between the logit model and the MCI model.

MDI and SANDDABS

SANDDABS is a model which has been used for more than fifteen years at the Market Research Corporation of America for estimating shifts in market size and brand preferences. A SANDDABS analysis begins at the level of the individual household. This stage of the analysis can be represented by the tableau below (which may contain any number of rows and columns):

$$
\begin{array}{c|c|c|c|c}
 & A & B & C & \underline{\text{Period I}} \\
\hline
A & \delta_{AA} & \delta_{AB} & \delta_{AC} & P_A^1 \\
\hline
B & \delta_{BA} & \delta_{BB} & \delta_{BC} & P_B^1 \\
\hline
C & \delta_{CA} & \delta_{CB} & \delta_{CC} & P_C^1 \\
\hline
\underline{\text{Period II}}: & P_A^2 & P_B^2 & P_B^2 &
\end{array}
$$

$$\sum_i P_i^1 = \sum_i P_i^2$$

The margins of the tableau usually represent the volumes of brands A, B, C,... purchased by a given household in two periods of equal length (although they may for different purposes represent units purchased, purchase/non-purchase indicators, or other units). In practice, rows and columns are added to the tableau to reflect changes in the total category volume sold. In this way, SANDDABS can allocate brand shifting volumes to market size change, and vice versa; and the sum of the tableau row sums equals the sum of the column sums.

A household's repeat purchase of a brand is reasonably the minimum of its period I and period II volumes, eg.

$$\delta_{AA} = \min \left\{ P_A^1, P_A^2 \right\} .$$

Shifting volumes are then computed from reduced margins

$$\hat{P}_i^1 = P_i^1 - \delta_{ii}$$

$$\hat{P}_i^2 = P_i^2 - \delta_{ii} \qquad \text{for all brands } i$$

as follows:

$$\delta_{ij} = \hat{P}_i^1 \hat{P}_i^2 / \sum_i \hat{P}_i^1 \qquad \text{for all } i \neq j.$$

The resulting tableaux are summed over all households to obtain the estimate of the total market's behavior.

Charnes, Cooper and Learner (1978) showed that an individual household's brand shifting preferences, traditionally computed as above in the SANDDABS model, are the solution of the MDI problem

$$\text{Max} - \sum_{i \neq j} \delta_{ij} \ln \left[\frac{\delta_{ij}}{e / \sum_i \hat{P}_i^1} \right]$$

$$\text{subject to} \quad \sum_{\substack{j \\ i \neq j}} \delta_{ij} = \hat{P}_i^2$$

$$\sum_{\substack{i \\ i \neq j}} \delta_{ij} = \hat{P}_j^1$$

$$\text{all } \delta_{ij} \geq 0$$

The proof is based on a demonstration that the traditional solution $\delta_{ij} = \hat{P}_i^1 \hat{P}_j^2 / \sum_i \hat{P}_i^1$ produces equal values for the primal and dual MDI functionals. This is a sufficient condition for optimality.

Phillips (1978) set down the asymptotic distribution for the Kullback information number associated with the SANDDABS summary matrix (the sum of all household matrices).

These developments showed first of all that the SANDDABS procedure had implicit "underlying optimizations", and secondly that SANDDABS could be used as a flexible hypothesis testing tool, using the asymptotic theory of the associated MDI number.

SANDDABS thus comprehends the ability to constructively test issues of general interest in marketing:

--Are brand shares stationary over a given period of time?

--Have switching patterns changed over time?

--Is switching proportional to brand share?

--Is a given brand partitioning scheme statistically valid?

--Is a given consumer segmentation scheme statistically valid?

See Charnes, Cooper, Learner and Phillips (1980) and Learner and Phillips (1979) for further discussion and examples.

MDI and the Hendry Model

The Hendry system is a proprietary and little-understood set of models for marketing management, developed by the Hendry Corporation. Few technical writeups have been released (see Hendry (1971)), but some items concerning the Hendry market segmentation model seem fairly certain:

(1) The model is based on a combinatorial definition of "entropy".

(2) Hierarchical brand attribute structures are posited and taken to correspond to a hierarchical decision process on the part of the consumer.

(3) The latter structure is not statistically tested; in fact, all of "Hendrodynamics" has a markedly deductive flavor, but from subjective postulates.

(4) Heavy emphasis is placed on a scalar "switching constant" which measures "intensity of competition" between brands.

(5) Brand switching volumes (pairwise) are assumed to be proportional to the product of the brands' shares, in an "equilibrium" situation.

The information theoretic marketing models developed by the current authors have been detailed elsewhere (Charnes, Cooper and Learner (1978); Phillips (1978); and Charnes, Cooper, Learner and Phillips (1980)). In addressing the Hendry models, we begin by reiterating the superior flexibility of MDI over max-entropy for representing estimated quantities relative to a hypothesized or baseline state of affairs. Further, the flexible hypothesis testing capability of the MDI method allows goodness-of-fit tests of the market segments and structures suggested by Hendry, among others.

Charnes, Cooper, Learner and Phillips (1980) reinterpreted the Hendry switching constant as a vulnerability ratio, and provided a method for simultaneous determination of the ratios (in contrast to the trial-and-error method given in Kalwani and Morrison's (1977) interpretation of Hendry). Following this, the same authors developed an information theoretic test for the validity of a product segmentation based on the vulnerability ratio.

This was the first statistical test known to the authors of any of the consequences of the Hendry approach. The test involved characterizing the apparent segmentation (given by the Charnes-Cooper-Learner-Phillips algorithm (1980)) by a set of linear constraints on a variable switching matrix $[p_{ij}]$. Given an observed switching matrix $[q_{ij}]$, minimizing $I(p:q)$ subject to the constraints constituted a test of whether the segment structure was consistent with the observed switching. The test will be detailed in a future report. *

*Charnes, Cooper, Learner, and Phillips (1980a).

Further Applications of MDI in Marketing

A recent article of Jones and Zufryden (1978) combined a logit model of brand choice with a negative binomial purchase frequency hypothesis to produce a components-of-sales model for consumer package goods. The bulky parameter estimation apparatus attached to this model shows marked contrast to the simultaneous and easy solution of the MDI estimate--for which the logit representation is a built-in consequence.

The information theoretic "MARK-IT" model (Phillips (1978)) estimates the joint distribution of three components of brand sales within a product category: brand loyalty (i.e., probability of brand choice), purchase frequency, and transaction size (lbs.). In the original development of MARK-IT, the greatest emphasis was given to management interpretation of the model; however, the present work makes it clear that a logit model of brand choice can follow directly from MARK-IT, and that the capability resides in MARK-IT for testing any distributional hypothesis concerning transaction frequency or transaction size.

Tests of other marketing questions (including other aspects of the Hendry system, etc.), seem readily possible with MDI procedures. Charnes, Cooper, Learner and Phillips (1980) bring forth that the large-sample multinormality of multinomial brand purchase proportions should result in the appearance of "switching proportional to brand share" whether or not "equilibrium" is present. With additional data such as SANDDABS (described in an earlier section) one can test hypotheses by MDI as tests in contingency tables (Gokhale and Kullback (1978)).

See also Learner and Phillips (1979) for an exposition of MDI models which stresses managerial and decision theoretic issues.

Gokhale and Kullback (1978) explain procedures for testing nested hypotheses with MDI, and Phillips (1980) provides additional examples. Nested hypotheses are effected by adding or removing constraints in the canonical MDI problem; the associated information values and degrees of freedom are additive and can be displayed in an "Analysis of Information" table. The sequential procedure is both convenient and meaningful--as mentioned earlier, I(p:q) is a general distance measure and not merely a test statistic, and so even when an hypothesis is rejected, the MDI procedure will determine the best alternative hypothesis.

Conclusion

We have now covered a variety of topics in sufficient detail to suggest how the information statistic--including its extensions to constrained optimizations--could be used to unify different parts of market research. This includes topical areas such as brand switching and individual choice models as well as location and traffic flows. For methodological unification we have shown how such topics as logit and probit analysis, with corresponding regressions, can be accorded information theoretic interpretations and uses. Other points of contact were also indicated with topics such as market segmentation and preference analysis and, of course, still others could also be developed in detail and the same applies to other methodologies besides the ones covered in this paper.

Much remains to be done, of course, in identifying limits to the unifying power of these approaches as well as in establishing more rigorously the contacts we have already indicated. En route to the indicated unification, we should also be able to benefit from improved abilities to deal with different classes of marketing problems.

Particular attention is called to the joining of mathematical programming to information theory which was illustrated in contexts such as brand switching and consumer choice analysis. We did not, however, examine the possible further uses of these extensions to comprehensive market planning and policy and control evaluations. Even within the limits of the separation of statistical analyses from managerial planning models that have been customary in marketing--but not in mathematical programming--

we have also indicated some new possibilities. One of these involves the possible use of constraints to deal with issues such as the assumed "independence of irrelevant alternatives" that has proved awkward to treat in other approaches such as multi-dimensional scaling. We also indicated how the problem of statistically testing nested hypotheses can be treated by constraint adjunction and elimination. The hierarchical marketing structures of the marketing literature (Kalwani (1979), Urban, Johnson and Brudnick (1979)) can also be treated similarly and tested step by step instead of being only subjectively posited and re-posited as at present.

These constraint possibilities, on the other hand, raise new problems for statistical research and for mathematical (computational) research as well.

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13. ABSTRACT

Constrained minimum discrimination information methods provide a basis for a unified approach to a wide range of problems in marketing research. For instance, they lead to characterization parallel to those of the Hendry system and other entropic approaches, with greater economy of assumptions. Goodness-of-fit tests and a structure for decision modelling are supplied from the same basic models with a range of applications that include market segmentation and brand shifting choices. Other probabilistic models of marketing choice (logit, MCI, etc.) are also comprehended in ways that resolve many logical and computational difficulties in these other approaches.

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